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PARTIAL DIFFERENTIAL EQUATIONS WITH MATRICIAL COEFFICIENTS AND GENERALIZED TRANSLATION OPERATORS

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Abstract. Let Δ_α be the Bessel operator with matricial coefficients defined on $(0,\infty)$ by

$$\Delta_\alpha U(t) = U''(t) + \frac{2\alpha + I}{t} U'(t)$$

where α is a diagonal matrix and let q be an $n \times n$ matrix-valued function. In this work, we prove that there exists an isomorphism X on the space of even $\mathcal{C}^\infty$, $\mathbb{C}^n$-valued functions which transmutes Δ_α and $(\Delta_\alpha + q)$. This allows us to define generalized translation operators and to develop harmonic analysis associated with $(\Delta_\alpha + q)$. By use of the Riemann method, we provide an integral representation and we deduce more precise information on these operators.

1. Introduction

Let $(\Delta_\alpha + q)$ be the perturbed Bessel operator with matricial coefficients defined, on $]\ 0, \infty[$, by

$$(\Delta_\alpha + q)U(t) = U''(t) + \frac{2\alpha + I}{t} U'(t) + q(t)U(t), \tag{1.1}$$

where U and q are matrix-valued functions, and α denotes the n order diagonal matrix

$$\alpha = \begin{bmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_n \end{bmatrix},$$

with $\alpha_n \geq ... \geq \alpha_1 > -\frac{1}{2}$.

It is well known (see [20]) that the radial Schrödinger equation with coupling between l^{th} and the $(l+2)^{nd}$ angular momentum gives us a system of singular equations which can be associated with an operator in the form (1.1).

The first goal of this paper is to establish the existence of an operator X (transmutation operator) which transmutes $(\Delta_\alpha + q)$ and Δ_α, in the sense

$$(\Delta_\alpha + q)X = X\Delta_\alpha,$$

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and then reduces the study of $(\Delta_\alpha+q)$ to that of Δ_α. This will be done thanks to an integral representation of the regular solution of the following differential equation:

$$U''(t) + \frac{I+2\alpha}{t}U'(t) + q(t)U(t) + \lambda^2 U(t) = 0, \qquad \lambda \in \mathbb{C}, \tag{1.2}$$

in terms of Bessel functions.

The second part is devoted to the study of generalized translation operators $T_y,\ y \in \mathbb{R}$, associated with $(\Delta_\alpha + q)$. This follows the framework of J. L. Lions [16] and generalizes some results known in the scalar case (see [6] and [22]). We use two techniques to study these operators; the first one is the Fourier transform associated with the spectral decomposition of $(\Delta_\alpha+q)$. We will see that if f is a $\mathcal{C}^\infty$ matrix-valued function with compact support, then the function $u(x,y) = T_y f(x)$ is a solution of the Cauchy problem

$$\begin{cases} (\Delta_\alpha + q)_x u(x,y) = (\Delta_\alpha + q)^*_y u(x,y), \\ u(x,0) = f(x),\ \left.\dfrac{\partial u}{\partial y}\right|_{(x,0)} = 0, \end{cases} \tag{1.3}$$

where

$$(\Delta_\alpha + q)^*\ U(t) = U''(t) + U'(t)\frac{2\alpha+I}{t} + U(t)q^*(t),$$

and $q^*(t)$ is the conjugate of the matrix $q(t)$.

This leads to the second technique which consists in solving (1.3). The Riemann method seems to be convenient for our purpose. Combining these two ways we deduce some properties of the generalized translation operators. Among others we deduce the product formula for the regular solution of (1.2).

The Riemann method has been used by B. L. J. Braaksma and H. S. V. De Snoo and others (see [2], [5], [7]) to solve the scalar Cauchy problem associated with (1.3), and by Coz (see [10]) to study the kernel of the transmutation operator associated with a singular second order differential operator with matricial coefficients.

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2. Notation and hypotheses

Let $M_n(\mathbb{C})$ be the space of square n order complex matrices and I the identity matrix. The conjugate of a matrix A in $M_n(\mathbb{C})$ is denoted by A^*. The norm of $A = (a_{jk})_{1\le j,k\le n}$ will be defined by

$$||A|| = \max_{1\le j\le n} \sum_{1\le k\le n} |a_{jk}|.$$

Let f be a holomorphic matrix-valued function. For any bounded domain $D \subset \mathbb{C}$ we denote the norm of f on D by

$$||f||_D = \sup_{z\in D} ||f(z)||.$$

Let J_ν and $H^{(1)}_\nu$ be the Bessel functions of the first and the third kind of index $\nu,\ \nu \in \mathbb{R}$ (see [22]). We denote by $\Gamma(\alpha), J_\alpha$ and $H^{(1)}_\alpha$ the diagonal matrix-valued functions defined for, $1 \le k \le n$, by

$$[\Gamma(\alpha)]_{kk} = \Gamma(\alpha_k), \quad [J_\alpha]_{kk} = J_{\alpha_k}, \quad \text{and} \quad [H^{(1)}_\alpha]_{kk} = H^{(1)}_{\alpha_k}. \tag{2.1}$$

Let $\mathcal{J}_{\alpha+p},\ p \geq 0$, and j_α be the diagonal matrix-valued functions defined, for $t > 0$ and $\lambda \in \mathbb{C}$, by

$$\mathcal{J}_{\alpha+p}(\lambda, t) = 2^\alpha \Gamma(\alpha + I)\lambda^{-\alpha-pI} J_{\alpha+pI}(\lambda t),\ j_\alpha(\lambda t) = t^{-\alpha}\mathcal{J}_\alpha(\lambda, t). \tag{2.2}$$

Let $G(t)$ be the $M_n(\mathbb{C})$-valued function defined on $[\,0, \infty[$ by

$$G(t) = \begin{bmatrix} t^{\alpha_1+\frac{1}{2}} & & 0 \\ & \ddots & \\ 0 & & t^{\alpha_n+\frac{1}{2}} \end{bmatrix},$$

so the operator $(\Delta_\alpha + q)$ can be written in the form

$$(\Delta_\alpha + q)U(t) = G^{-1}(t)[GU']'(t) + q(t)U(t).$$

The transformation $v = GU$ carries the operator $(\Delta_\alpha + q)$ into the perturbed Bessel operator L^Q given by

$$L^Q v(t) = v''(t) + [\frac{\frac{I}{4} - \alpha^2}{t^2} + Q(t)]v(t), \qquad t > 0.$$

The notions of continuity, differentiability, etc., when applied to matrix-valued functions, are always meant to hold for each coefficient. For Y and Z two differentiable matrix functions, we denote by $\mathcal{W}[Y, Z]$ their Wronskian

$$\mathcal{W}[Y, Z](t) = Y^*(t)G^2(t)Z'(t) - Y'^*(t)G^2(t)Z(t), \tag{2.3}$$

which is independent of t if both Y and Z satisfy (1.2).

We need the following hypotheses:

$(\mathbf{A_1})$ q is even and holomorphic on $\mathbb{C}$.

$(\mathbf{A_2})$ $Q(t) = [GqG^{-1}](t),\ t > 0$, is an hermitian matrix.

$(\mathbf{A_3})$ $\sigma(t) = \displaystyle\int_t^\infty s||Q(s)||ds < +\infty,\ t > 0.$

3. Asymptotic behavior

3.1. Behavior at zero. We consider the class of singular second order differential equations

$$L^Q v(t) + \lambda^2 v(t) = 0, \qquad t > 0, \tag{3.1}$$

where λ is a real or complex parameter.

Thanks to an idea introduced by [13] (see also [18]) we show that the series

$$\Phi_\lambda(t) = G^{-1}(t)[\sum_{p\geq 0} t^{p+\frac{1}{2}} B_p(t)\mathcal{J}_{\alpha+p}(\lambda, t)]G(t) \tag{3.2}$$

is a formal solution of (3.1) provided that the coefficients $B_p,\ p \geq 0$, satisfy the recursive process

$$\begin{cases} [G^{-1}(t)B_0(t)G(t)]' & = & 0, \\ \left[t^{p+1}G^{-1}(t)B_{p+1}(t)G(t)\right]' & = & -\frac{t^p}{2}G^{-1}(t)[(\Delta_{-\alpha} + q^*)B_p](t)G(t). \end{cases}$$

Let us choose $B_0(t) = I$. Our purpose is to define $B_p, p \geq 1$, by the same mean as in [18]; we need the following lemma.

Lemma 3.1. *Under the hypotheses* (A_1) *and* (A_2)*, for* $p \in \mathbb{N}$*, the matrix functions* $B_{p+1}(t)$ *and* $t^{2p}G^{-2}(t)B_{p+1}(t)G^2(t)$*, where* B_{p+1} *is defined by*

$$B_{p+1}(t) = -\frac{1}{2t^{p+1}}G(t)\int_0^t u^p G^{-1}(u)\left[(\Delta_{-\alpha} + q^*)B_p\right](u)G(u)du G^{-1}(t), \tag{3.3}$$

are even and analytic.

Proof. From the hypotheses (A_1) and (A_2) we deduce that, for $1 \leq i, j \leq n$, we have $\alpha_i - \alpha_j \in \mathbb{Z}$. The hypothesis (A_2) can be expressed in the form

$$u^{-\alpha_i+\alpha_j}\bar{q}_{ji}(u) = u^{\alpha_i-\alpha_j}q_{ij}(u);$$

then we define B_1 by

$$\begin{aligned}[B_1]_{ij}(t) &= -\frac{1}{2t^{-\alpha_i+\alpha_j+1}}\int_0^t u^{-\alpha_i+\alpha_j}\bar{q}_{ji}(u)du, \qquad i \leq j,\\ &= -\frac{1}{2t^{-\alpha_i+\alpha_j+1}}\int_0^t u^{\alpha_i-\alpha_j}q_{ij}(u)du, \qquad i \geq j.\end{aligned}$$

We deduce that the matrix functions $B_1(t)$ and $G^{-2}(t)B_1(t)G^2(t)$ are even and analytic and recursively we obtain the lemma. □

Proposition 3.2. *Suppose that* q *is holomorphic in the disc* $D(0, R)$*. Then*

i) *For* $k, p \geq 0$*, the functions* B_p*, defined by the recursive formula* (3.3)*, satisfy*

$$||B^{(k)}_{p+p_0}||_{D(0,R)} \leq \frac{k!p^k}{2^{p-1}R^k p!}\left[\frac{2cp^2}{R^2} + M\right]^{p-1}||B_{p_0}||_{D(0,R)} \tag{3.4}$$

where $c = ||I - 2\alpha||$*,* $M = ||q||_{D(0,R)}$ *and* $p_0 = \alpha_n - \alpha_1$*.*

ii) *The infinite series* (3.2) *is uniformly convergent on every compact subinterval of* $]0, 1 + ||I - 2\alpha||^{-1/2}Re^{-1}[$*; its sum* $\Phi_\lambda(t)$ *is an* n *order matrix whose rows are vector solutions of* (3.1)*, and which satisfies the asymptotic relation*

$$\Phi_\lambda(t) = G(t)[I + o(1)], \qquad \lim_{t\to 0^+} o(1) = 0. \tag{3.5}$$

Proof. Since (3.3) can be written as

$$[B_{p+1}]_{ij}(t) = -\frac{1}{2t^{\alpha_j-\alpha_i+p+1}}\int_0^t u^{\alpha_j-\alpha_i+p}[(\Delta_{-\alpha} + q^*)B_p]_{ij}(u)du, \quad 1 \leq i, j \leq n,$$

then, for $p + \alpha_j - \alpha_i \geq 0$, we use estimates of [18, p. 264], and the proposition is deduced. □

Corollary 3.3. *The equation* (1.1) *has an unique solution* $\psi(\lambda, t) = G^{-1}(t)\Phi_\lambda(t)$ *satisfying the following asymptotic behavior*:

$$\lim_{t\to 0}\psi(\lambda, t) = I.$$

Using the Sonine integral representation of the Bessel function of the first kind J_α (see [22]), we deduce:

Corollary 3.4. *The function* ψ *has the Sonine type integral representation*

$$\psi(\lambda, t) = j_\alpha(\lambda t) + \int_0^t M(t, u)G^2(u)j_\alpha(\lambda u)du, \tag{3.6}$$

where j_α *is given by* (2.1) *and*

$$M(t,u) = tG^{-2}(t)\sum_{1}^{\infty}\frac{B_p(t)(t^2-u^2)^{p-1}}{2^{p-1}\Gamma(p)}, \tag{3.7}$$

the series being uniformly convergent on every compact subinterval of $(0,\infty)$.

3.2. Behavior at infinity. Let us denote by $\mathcal{H}_\alpha^{(1)}$ the diagonal matrix defined by

$$\mathcal{H}_\alpha^{(1)}(t) = \sqrt{\frac{\pi t}{2}}e^{i(\alpha+\frac{I}{2})\frac{\pi}{2}}H_\alpha^{(1)}(t), \tag{3.8}$$

where H_α^1 is given by the formula (2.1).

Theorem 3.5. i) *For* $\lambda \in \mathbb{C}, \lambda \neq 0, t > 0$, *the differential equation* (3.1) *has a fundamental system of solutions* $F(\lambda,t)$ *and* $F(-\lambda,t)$ *satisfying the following asymptotic behavior*:

$$F(\lambda,t) = e^{-i\lambda t}[I + o(\frac{1}{|\lambda|t})], \qquad \lim_{|\lambda|t\to+\infty} o(\frac{1}{|\lambda|t}) = 0.$$

ii) *The mapping* $\lambda \mapsto F(\lambda,t)$ *is analytic in* $\{\lambda\in\mathbb{C}, \Im m\lambda < 0\}$ *and continuous in* $\{\lambda\in\mathbb{C}, \Im m\lambda \leq 0\}$.

Proof. Using the method of variation of parameters in the equation

$$v''(t) + \frac{\frac{I}{4}-\alpha^2}{t^2}v(t) + \lambda^2 v(t) = -Q(t)v(t), \qquad t > 0,$$

we obtain the following integral equation:

$$F(\lambda,t) = \mathcal{H}_\alpha^{(1)}(\lambda t) - \int_t^\infty K(\lambda,t,t')Q(t')F(\lambda,t')dt',$$

where the kernel $K(\lambda,t,t')$ is defined by

$$K(\lambda,t,t') = \frac{e^{2i\alpha\pi}}{2i\lambda}\Big[\mathcal{H}_\alpha^{(1)}(\lambda t')\mathcal{H}_\alpha^{(1)}(-\lambda t) - \mathcal{H}_\alpha^{(1)}(\lambda t)\mathcal{H}_\alpha^{(1)}(-\lambda t')\Big].$$

Using estimates on $\mathcal{H}_\alpha^{(1)}$, the hypothesis (A_3) and successive approximations we deduce the existence of $F(\lambda,t)$ and its behavior at infinity (see similar results in [4], [20]). □

Corollary 3.6. i) *Let* $0 < t_0 \leq t$; *for* $\lambda\in\mathbb{C}$, $\Im m\lambda \leq 0$, *and* $|\lambda| \geq \lambda_0 > 0$, *we have*

$$F'(\lambda,t) = -i\lambda e^{-i\lambda t}[I + o(\frac{1}{|\lambda|t})], \qquad \lim_{|\lambda|t\to+\infty} o(\frac{1}{|\lambda|t}) = 0.$$

ii) *For* $\Im m\lambda < 0$, *we have*

$$\lim_{|\lambda|t\to+\infty}\dot{F}(\lambda,t) = 0, \qquad \lim_{|\lambda|t\to+\infty}\dot{F}'(\lambda,t) = 0.$$

3.3. The c-matrix. Let $E(\lambda,t)=G^{-1}(t)F(\lambda,t)$ be the solution of (1.2) associated with $F(\lambda,t)$. Using the known fact that the Wronskian of $\psi(\lambda,t)$ and $E(-\bar{\lambda},t)$ is independent of the variable t, we set

$$c(\lambda)=\frac{1}{2i\lambda}\mathcal{W}[E(-\bar{\lambda},t),\psi(\lambda,t)],\qquad \lambda\neq 0. \tag{3.9}$$

This matrix, the analogue of the Harish-Chandra c-function, intervenes in the density of the spectral measure associated with the operator $(\Delta_\alpha+q)$. The properties of $\psi(\lambda,t)$ and $E(\lambda,t)$ allow us to deduce the following results.

Proposition 3.7. i) *The mapping $\lambda\mapsto c(\lambda)$ is analytic for $\Im m\lambda<0$, and*

$$c(\lambda)=\sqrt{\frac{1}{2\pi}}2^\alpha\Gamma(\alpha)e^{-i(\alpha+\frac{I}{2})\frac{\pi}{2}}G^{-1}(\lambda)[I+o(\frac{1}{|\lambda|})],\quad \lim_{|\lambda|\to\infty}o(\frac{1}{|\lambda|})=0. \tag{3.10}$$

ii) *There exists a constant $N>0$ such that, for any $\lambda\in\mathbb{C}$, $\Im m\lambda\leq 0$, $|\lambda|\geq N; c(\lambda)$ is invertible. The mapping $\lambda\mapsto c^{-1}(\lambda)$ is a meromorphic function on $\Im m\lambda<0$ with simple poles.*

Proof for similar results are given in [17].

4. Fourier transform

Let $\mathcal{D}$ be the space of even $\mathcal{C}^\infty,\mathbb{C}^n$-valued functions with compact support.

Definition 4.1. For every f in $\mathcal{D}$ we define the generalized Fourier transform $\mathcal{F}(f)$, associated with $(\Delta_\alpha+q)$, by

$$\mathcal{F}(f)(\lambda)=\int_0^\infty\psi^*(-\bar{\lambda},t)G^2(t)f(t)dt. \tag{4.1}$$

Remark 4.2. The Fourier-Bessel transform $\mathcal{F}_\alpha$, associated with Δ_α, is a $\mathbb{C}^n$-valued operator defined for f $\in\mathcal{D}$ by

$$\{\mathcal{F}_\alpha(f)\}_k(\lambda)=\int_0^\infty j_{\alpha_k}(\lambda u)u^{2\alpha_k+1}f_k(u)du,\qquad 1\leq k\leq n.$$

Let L_2^G be the Hilbert space defined by

$$L_2^G=\{f:]0,\infty[\to\mathbb{C}^n\mid ||f||_2^2=\int_0^\infty f^*(x)G^2(x)f(x)dx<+\infty\}.$$

The Kato-Rellich criterion shows that the operator $(\Delta_\alpha+q,\mathcal{D})$ is an unbounded essentially self adjoint operator in L_2^G. Its absolute continuous spectrum, parametrized by λ, is equal to $[0,\infty[$. The discrete spectrum is composed with a finite number of negative eigenvalues: $-\lambda_1^2,...,-\lambda_m^2$, where the $\lambda_j, 1\leq j\leq m$, are zeros of $\lambda\to detc(\lambda)$. Let R_j be the residue of $c^{-1}(\lambda)$ at λ_j; then $\psi(\lambda_j,t)R_j$ is the associated eigenfunction.

Let S_j be the inverse matrix of

$$\int_0^\infty[\psi(\lambda_j,t)R_j]^*G^2(t)\psi(\lambda_j,t)R_jdt, \tag{4.2}$$

and let S be the matrix defined, for $\lambda\neq\lambda_j$, by

$$S^{-1}(\lambda)=2\pi c^*(\lambda)c(\lambda). \tag{4.3}$$

The properties of $c(\lambda)$ and its asymptotic behavior allow us to deduce that $S(\lambda)$ is a tempered measure and that

$$S(\lambda) = S^*(\lambda) = S(-\lambda). \tag{4.4}$$

To obtain the spectral theorem we shall introduce some new spaces (see [12]). Let L_2^S be the Hilbert space of $\mathbb{C}^n$-valued functions f defined by

$$L_2^S = \{f:\ ]0,\infty[\to \mathbb{C}^n \mid ||f||_S^2 = \int_0^\infty f^*(\lambda)S(\lambda)f(\lambda)d\lambda < +\infty\}. \tag{4.5}$$

For u and v two elements of $(\mathbb{C}^n)^m$ we put

$$\langle\langle u, v\rangle\rangle = \sum_{j=1}^m v_j^* S_j u_j.$$

Finally let $\widehat{\mathcal{H}} = L_2^S \oplus (\mathbb{C}^n)^m$, and

$$H = \{g : \mathbb{C} \to \mathbb{C}^n,\ even,\ analytic \mid \exists R > 0, \forall k \in \mathbb{N},$$
$$sup_{\lambda\in\mathbb{C}}[(1 + |\lambda|^2)^k e^{-R|\Im m\lambda|}||g(\lambda)|| < +\infty\}.$$

Making use of the spectral theory of self adjoint operators on Hilbert spaces (see [11]) and the last results we deduce the following

Theorem 4.3. i) *(Inversion Formula) The mapping $f \mapsto \mathcal{F}(f)$ extends to an isometric isomorphism from L_2^G onto $\widehat{\mathcal{H}}$; the inverse mapping is given by*

$$f(t) = \int_0^\infty \psi(\lambda, t)S(\lambda)\mathcal{F}(f)(\lambda)d\lambda \ + \ \sum_{j=1}^m \psi(\lambda_j, t)R_j S_j \hat{f}_j, \tag{4.6}$$

with

$$\hat{f}_j = \int_0^\infty [\psi(\lambda_j, s)R_j]^* G^2(s)f(s)ds.$$

ii) *(Plancherel Formula) For any f in L_2^G we have*

$$||f||_2^2 = ||\mathcal{F}(f)||_S^2 + \sum_{j=1}^m \hat{f}_j^* S_j \hat{f}_j. \tag{4.7}$$

iii) *(Paley-Wiener type theorem) The mapping $f \mapsto \mathcal{F}(f)$ is a bijection from $\mathcal{D}$ onto H.*

5. Transmutation

Let C^2 be the space of twice continuous differentiable $\mathbb{C}^n$-valued functions defined on $]0,\infty[$. Let C_* be the space of even $C^\infty, \mathbb{C}^n$-valued functions, equipped with the topology of the uniform convergence, on every compact subset of $\mathbb{R}$, of the functions and their derivatives. Let $\mathcal{E}$ be the subspace of functions in C^2 such that $G^{-1}f \in C_*$.

Let us recall (see [18]) that the operator $\mathcal{X}$ defined on $\mathcal{E}$ by

$$\mathcal{X}f(t) = f(t) + G(t)\int_0^t M(t, u)G(u)f(u)du, \qquad t > 0, \tag{5.1}$$

where $M(t,u)$ is given by the formula (3.7), is a permutation operator between L^Q and L^0, that is,

$$L^Q\mathcal{X} = \mathcal{X}L^0. \tag{5.2}$$

5.1. Transmutation associated with $(\Delta_\alpha + q)$. Since the operator $(\Delta_\alpha + q)$ is related to L^Q by the formula

$$(\Delta_\alpha + q) = G^{-1}L^Q G, \tag{5.3}$$

let us consider the operator X defined by

$$X = G^{-1}\mathcal{X}G; \tag{5.4}$$

then the formulas (5.1) and (5.2) allow us to deduce that for any h in $\mathcal{D}$

$$(\Delta_\alpha + q)Xh = X\Delta_\alpha h,$$

with

$$Xh(t) = h(t) + \int_0^t M(t,u)G^2(u)h(u)du, \qquad t > 0. \tag{5.5}$$

Theorem 5.1. *The operator X is an isomorphism on the space C_* and for any h in C_* we have*

i) $\lim_{t\to 0} Xh(t) = h(0)$,
ii) $X\Delta_\alpha h = (\Delta_\alpha + q)Xh$.

Proof. Let us consider the integral equation of Gelfand-Levitan type

$$g(x) = h(x) + \int_0^x M(x,u)G^2(u)h(u)du, \qquad x > 0.$$

Lemma (3.1) and Proposition 3.2 lead that the kernel $M(x,u)G^2(u)$ is continuous for $0 \le u \le x$, so the previous integral equation has an unique solution which is continuous. To deduce i) we note, using the Proposition 3.2 that

$$Xh(t) = h(t) + \sum_0^\infty \frac{G^{-2}(t)t^{2p+2}B_{p+1}(t)G^2(t)}{2^p p!} \int_0^1 (1-v^2)^p G^2(v)h(tv)dv.$$

To have ii) we take $f = Gh$, $h \in C_*$ and we use the formulas (5.2), (5.3) and (5.4). □

Definition 5.2. We denote by C'_* the space of even $\mathbb{C}^n$ valued distributions with compact support.

Proposition 5.3. *The operator ${}^tX$ defined on C'_* by*

$$\langle {}^tXT, g\rangle = \langle T, Xg\rangle, \qquad g \in C_*, \tag{5.6}$$

is an isomorphism on C'_*.

Proof. It is an immediate consequence of Theorem 5.1. □

Corollary 5.4. *For any g in $\mathcal{D}$, ${}^tX(g)$ is given by*

$${}^tXg(x) = g(x) + \int_x^\infty M^*(u,x)G^2(u)g(u)du, \quad x > 0. \tag{5.7}$$

The function ${}^tX(g)$ is even and continuous with compact support.

5.2. Generalized Fourier transform and transmutation.

Lemma 5.5. *For any f in $\mathcal{D}$, Xf is in $\mathcal{F}^{-1}(L_2^S)$.*

Proof. For f in $\mathcal{D}$, we have the following Fourier-Bessel inverse formula, associated with Δ_α

$$f(t) = \frac{1}{2\pi}\int_0^\infty j_\alpha(t\lambda)G^2(\lambda)\mathcal{F}_\alpha f(\lambda)d\lambda,$$

where j_α is given by (2.2). Using (3.6) and the last formula, we deduce that

$$\begin{aligned} Xf(t) &= \frac{1}{2\pi}\int_0^\infty \psi(\lambda,t)G^2(\lambda)\mathcal{F}_\alpha f(\lambda)d\lambda \\ &= \frac{1}{2\pi}\int_0^\infty \psi(\lambda,t)S(\lambda)[S^{-1}(\lambda)G^2(\lambda)\mathcal{F}_\alpha f(\lambda)]d\lambda, \end{aligned}$$

so we deduce the lemma. □

The following proposition allows us to link the operators $\mathcal{F}$ and $\mathcal{F}_\alpha$ using the transmutation operator X.

Proposition 5.6. *For any f in $\mathcal{D}$ we have*

$$\mathcal{F}Xf(\lambda) = c^*(\lambda)c(\lambda)G^2(\lambda)\mathcal{F}_\alpha f(\lambda), \qquad \lambda \in \mathbb{C}, \tag{5.8}$$

$$\mathcal{F}(f) = \mathcal{F}_\alpha \, {}^tX(f). \tag{5.9}$$

Proof. Formula (5.8) is a consequence of the previous lemma and (4.3).

The operators $\mathcal{F}$ and $\mathcal{F}_\alpha$ are isometries so that for f and g in $\mathcal{D}$ we have

$$\langle Xf, g\rangle_2 = \langle \mathcal{F}Xf, \mathcal{F}g\rangle_{\widehat{\mathcal{H}}},$$

and

$$\langle f, {}^tX(g)\rangle_2 = \langle \mathcal{F}_\alpha(f), \mathcal{F}_\alpha \, {}^tX(g)\rangle_2;$$

using the formula (5.8) we show that

$$\langle \mathcal{F}(Xf), \mathcal{F}(g)\rangle_{\widehat{\mathcal{H}}} = \langle \mathcal{F}_\alpha(f), \mathcal{F}(g)\rangle_2.$$

The density of $\mathcal{F}_\alpha(\mathcal{D})$ in L_2^G completes the proof of (5.9). □

Theorem 5.7. *The operator X is an isomorphism of L_2^G.*

Proof. Since $\mathcal{D}$ is dense in L_2^G, it suffices to have the result on $\mathcal{D}$. From Lemma 5.5 and the formula (5.8), we have (see [6])

$$\begin{aligned} ||X(f)||_2^2 &= ||\mathcal{F}X(f)||_S^2 = \frac{1}{2\pi}\int_0^\infty ||c(\lambda)G^2(\lambda)\mathcal{F}_\alpha f(\lambda)||^2 d\lambda \\ &\le \frac{1}{2\pi}\int_0^\infty ||c(\lambda)G(\lambda)||^2 ||G(\lambda)\mathcal{F}_\alpha f(\lambda)||^2 d\lambda. \end{aligned}$$

From (3.10) we deduce that $||c(\lambda)G(\lambda)||$ is bounded and so there exists a positive constant N_1 such that

$$||c(\lambda)G(\lambda)|| \le N_1.$$

Then we have

$$||X(f)||_2 \le N_1||\mathcal{F}_\alpha(f)||_2 \le N_1||f||_2.$$

Conversely, for f in $\mathcal{D}$, using (5.8) we have

$$\mathcal{F}_\alpha f(\lambda) = 2\pi G^{-2}(\lambda) S(\lambda) \mathcal{F} X f(\lambda);$$

we deduce that

$$||f||_2^2 = ||\mathcal{F}f||_S^2 = ||2\pi G^{-2}(\lambda) S(\lambda) \mathcal{F} f(\lambda)||_2^2$$

$$= \int_0^\infty ||2\pi G^{-1}(\lambda) S(\lambda) \mathcal{F} X f(\lambda)||^2 d\lambda.$$

Using again (3.10) we deduce that there exists a constant N_2 such that

$$||f||_2 \leq N_2 ||\mathcal{F}(Xf)||_S.$$

Finally the Plancherel formula allows us to deduce that

$$N_1^{-1} ||X(f)||_{\mathcal{H}} \leq ||f||_{\mathcal{H}} \leq N_2 ||X(f)||_{\mathcal{H}}, \qquad f \in \mathcal{D}.$$

This suffices to conclude the theorem. □

Remark 5.8. i) We define the generalized Fourier transform $\mathcal{F}$ and the transmutation operator X on the space $\tilde{\mathcal{D}}$ of C^∞, $M_n(\mathbb{C})$-valued functions, with compact support, respectively by the same expressions (4.1) and (5.5).

In particular Theorem 4.3, Propositions 5.3, 5.6 and Corollary 5.4 remain valid for these transformations.

ii) We denote by $\mathcal{L}_2$ the Hilbert space defined by

$$\mathcal{L}_2 = \{f :]0, \infty[\to M_n(\mathbb{C}) \mid |||f|||_2^2 = \sum_{i,j=1}^n \int_0^\infty f_{ij}^*(u) u^{2\alpha_i + 1} f_{ij}(u) du < \infty\}.$$

Theorem 5.7 remains valid for this space.

6. Generalized translations and convolution

Definition 6.1. i) We denote by T_y, $y \in \mathbb{R}$, the generalized translations operators, associated with $(\Delta_\alpha + q)$, and defined by

$$T_y f = \mathcal{F}^{-1}[\mathcal{F}(f)(\lambda) \psi^*(\bar{\lambda}, y)], \qquad f \in \tilde{\mathcal{D}}. \tag{6.1}$$

ii) Let f and g be in $\tilde{\mathcal{D}}$. The generalized convolution product, associated with $(\Delta_\alpha + q)$, is the function $f \sharp g$ defined by

$$f \sharp g = \mathcal{F}^{-1}[\mathcal{F}(f) \mathcal{F}(g)]. \tag{6.2}$$

If we use (4.6) and (6.1), then the previous definition takes the form

$$f \sharp g(x) = \int_0^\infty T_y f(x) G^2(y) g(y) dy. \tag{6.3}$$

Remark 6.2. i) In the particular case when $q = 0$ we denote by $\mathcal{T}_x$ (resp. $\star$) the generalized translation (resp. the convolution product) associated with Δ_α.

ii) When $g \in \tilde{\mathcal{D}}$ and $T \in \tilde{\mathcal{C}}'_*$ (the space of even matrix-valued distributions with compact support), we can define $T \star g$ by

$$T \star g = \mathcal{F}_\alpha^{-1}[\mathcal{F}_\alpha(T) \mathcal{F}_\alpha(g)].$$

It suffices to note that $\mathcal{F}_\alpha(T)$ belongs to the space of matrix-valued slowly increasing functions of exponential type.

The following properties derive immediately from Definition 6.1.

Properties 6.3. *For any $x \in \mathbb{R}$ and $f \in \tilde{\mathcal{D}}$, we have*

- $T_0 f(x) = f(x)$,
- $T_x(f) \in \tilde{\mathcal{D}}$,
- $(\Delta_\alpha + q)^*_x \; T_x f = T_x(\Delta_\alpha + q) f$,

where

$$(\Delta_\alpha + q)^* U(t) = U''(t) + U'(t)\frac{2\alpha + I}{t} + U(t) q^*(t).$$

Theorem 6.4. *For any f in $\tilde{\mathcal{D}}$, $(x, y) \in \mathbb{R} \times \mathbb{R}$, the function $u(x, y) := T_y f(x)$ is the unique solution of the Cauchy problem (1.3), given by*

$$\begin{cases} (\Delta_\alpha + q)_x u(x, y) = (\Delta_\alpha + q)^*_y u(x, y), \\ u(x, 0) = f(x), \; \left.\dfrac{\partial u}{\partial y}\right|_{(x,0)} = 0, \end{cases}$$

Proof. For f in $\tilde{\mathcal{D}}$, by definition, we have

$$T_y f(x) = \int_0^\infty \psi(x, \lambda) S(\lambda) \mathcal{F}(f)(\lambda) \psi^*(y, \lambda) d\lambda \; + \; \sum_{j=1}^m \psi(x, \lambda_j) R_j S_j \hat{f}_j \psi^*(y, \lambda_j);$$

then it is easy to see that $T_y f(x)$ is a solution of the problem (1.3). To have the unicity, we apply the Fourier transform, with respect to x, for the two members of the equation

$$(\Delta_\alpha + q)_x \; u(x, y) = (\Delta_\alpha + q)^*_y u(x, y).$$

Then the function $v(\lambda, y) = \mathcal{F} u(., y)(\lambda)$ is a solution of the problem

$$(\Delta_\alpha + q)^*_y \; v(\lambda, y) = \lambda^2 v(\lambda, y),$$

which is a differential equation of Fuchs type with a regular condition at zero. The unicity of the solution of a such system completes the proof. □

The following proposition allows us to link the convolution products $\sharp$ and $\star$ by means of the transmutation operator X.

Proposition 6.5. *For f and g in $\tilde{\mathcal{D}}$, we have*

$$(Xf)\sharp g = X(f \star {}^t X g), \tag{6.4}$$

$${}^t X(f \sharp g) = {}^t X(f) \star {}^t X(g). \tag{6.5}$$

Proof. By the definition (6.2) we have

$$(Xf)\sharp g = \mathcal{F}^{-1}[\mathcal{F}(Xf)\mathcal{F}(g)].$$

The formulas (5.8) and (5.9) yield that

$$Xf \sharp g = \mathcal{F}^{-1}[c^*(\lambda) c(\lambda) G^2(\lambda) \mathcal{F}_\alpha(f \star {}^t X g)(\lambda)];$$

then we have (6.4) (from an idea of [1]).

Using the formulas (5.9) and (6.2), we have

$$\begin{aligned} (\mathcal{F}_\alpha \circ {}^t X)(f \sharp g) &= (\mathcal{F}_\alpha \circ {}^t X)(f)(\mathcal{F}_\alpha \circ {}^t X)(g) \\ &= \mathcal{F}_\alpha[{}^t X(f) \star {}^t X(g)]; \end{aligned}$$

the invertibility of $\mathcal{F}_\alpha$ allows us to deduce (6.5). □

In the proposition below we link the generalized translation operator T_x and $\mathcal{T}_x$ by means of the transmutation operator X.

Proposition 6.6. *For any f in $\tilde{\mathcal{D}}$ we have*

$$T_x f(y) = {}^tX_y^{-1}[X_x(\mathcal{T}_{(.)}{}^tXf)^*(y)]^*. \tag{6.6}$$

Proof. We note that (see [21])

$$\psi^*(\lambda, t) = \mathcal{F}_\alpha[K^*(t, .)](\lambda);$$

here $K(t, .)$ denotes the distribution, in $\mathcal{C}'_*$, defined by $\langle K(t, .), f\rangle = Xf(t)$, and Xf is given by the formula (5.5). Using Definition 6.1 we have

$$[\mathcal{F}_\alpha \circ {}^tX]T_x f = \mathcal{F}_\alpha[{}^tX(f)]\mathcal{F}_\alpha[K^*(x, .)] = \mathcal{F}_\alpha[{}^tX(f) \star K^*(x, .)],$$

or equivalently

$$T_x f(y) = {}^tX_y^{-1}[{}^tX(f) \star K^*(x, .)](y).$$

From the formula (6.5) we have

$$T_x f(y) = {}^tX_y^{-1} \int_0^x [\mathcal{T}_u({}^tXf)(y)]G^2(u)K^*(x, u)du,$$

so we deduce the formula (6.6). □

By use of Remark 5.8 and the previous proposition we have

Corollary 6.7. *The mapping $f \to T_y(f)$, $y \in \mathbb{R}$, extends to a continuous operator on $\mathcal{L}_2$.*

7. Riemann's method

In this section we recover an integral representation for the solution of the Cauchy problem (1.3). Estimations on the solution and the kernel of its integral representation are given. This allows us to deduce a product formula for the regular eigenfunction associated with $(\Delta_\alpha + q)$. For this we use the Riemann method, and we suppose that α is a scalar matrix. Since the problem (1.3) has no symmetry in x and y, Riemann's method leads us to study separately the cases $0 < x \leq y$ and $0 < y \leq x$. The Riemann function is used.

7.1. The Riemann function. Let Δ_{xy} denote the characteristic triangle in the (s, t) plane with vertices $P(x, y)$, $Q(x + y, 0)$ and $R(x - y, 0)$ (see Figure 1). B. L. J. Braaksma and H. S. V. De Snoo have studied in [2] the Riemann function in the domain Δ_{xy}. Since, in the scalar case, the problem (1.3) is symmetric in x and y, it is easy to extend the solution to the first quarter; this is not our case. A domain Ω of type Figure 2 is introduced to study the case $0 < x \leq y$. Estimations on the Riemann function are established in each case. we The Riemann function $\mathcal{R}$ is defined as the unique solution $v(s, t) = \mathcal{R}(x, y; s, t)$ of the characteristic boundary value problem

$$\begin{cases} (L_s^\alpha - L_t^\alpha)^{**}v(s, t) = 0, & (s, t) \in \Delta_{xy}, \\ v_s + v_t = (\alpha + \frac{1}{2})(\frac{1}{s} + \frac{1}{t})v, & s - t = x - y, \\ v_s - v_t = (\alpha + \frac{1}{2})(\frac{1}{s} - \frac{1}{t})v, & s + t = x + y, \\ v(x, y) = 1, \end{cases} \tag{7.1}$$

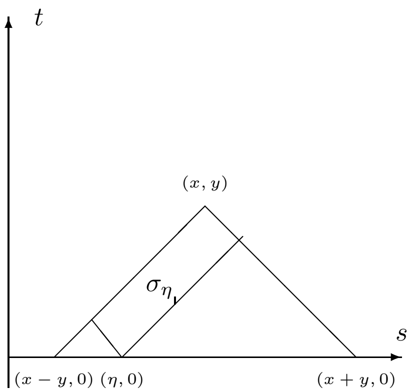

FIGURE 1. The case $0 < y \leq x$

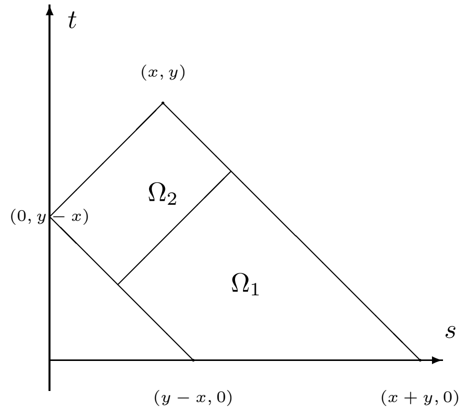

FIGURE 2. The case $0 < x \leq y$

where $(L_x^\alpha - L_y^\alpha)^{**}$ is the adjoint operator of $L_x^\alpha - L_y^\alpha$ and L_x^α is defined by

$$L_x^\alpha u = u_{xx} + \frac{2\alpha+1}{x} u_x, \ \alpha > -\frac{1}{2}.$$

This problem is dealt with as the scalar case (see [9]) and (7.1) has a solution which is a diagonal matrix given by

$$\mathcal{R}(x, y; s, t) = (\frac{st}{xy})^{\alpha+\frac{1}{2}} P_{\alpha-\frac{1}{2}}(1 - 2z)I,$$

where P_l denotes the l th -Legendre function and

$$z = \frac{\{(x-y)^2 - (s-t)^2\}\{(x+y)^2 - (s+t)^2\}}{16xyst}.$$

• In the case $0 < y \leq x$ we extend easily the results of [2] to have estimations on $\mathcal{R}$.

Let R be function defined by

$$R(x, y; s, t) = \begin{cases} \frac{st}{xy} & \text{if } \alpha \geq \frac{1}{2}, \\ \left(\frac{st}{xy}\right)^{\alpha+\frac{1}{2}} Z^{\frac{1}{2}-|\alpha|} & \text{if } 0 < |\alpha| < \frac{1}{2}, \\ \left(\frac{st}{xy}\right)^{\frac{1}{2}} Z^{\frac{1}{2}} \left(1 + Log Z^{-1}\right) & \text{if } \alpha = 0, \end{cases} \tag{7.2}$$

where

$$Z = 16xyst\{(x+y)^2 - (s-t)^2\}^{-1}\{(s+t)^2 - (x-y)^2\}^{-1}.$$

Then there exists a positive constant M_0 such that

$$||\mathcal{R}(x, y; s, t)|| \leq M_0 R(x, y; s, t), \ \forall (s, t) \in \Delta_{xy}. \tag{7.3}$$

Since $R(x, y; s, t)$ is bounded on Δ_{xy}, it follows that

$$||\mathcal{R}(x, y; s, t)|| \leq M_0', \ \forall (s, t) \in \Delta_{xy}. \tag{7.4}$$

• In the case $0 < x \le y$, one introduces the variables

$$2\xi = y - x, \qquad 2\eta = y + x,$$
$$2\xi_0 = t - s, \qquad 2\eta_0 = t + s.$$

Instead of z we use Chaundy's variables x_1 and x_2 (see [10]) defined as follows:

$$x_1 = \frac{(x+y-s-t)(y-x-t+s)}{4xs}, \quad x_2 = \frac{(x+y-s-t)(x-y-s+t)}{4yt},$$

so we have

$$1 - 2z = 1 - 2x_1 - 2x_2 - 2x_1x_2 = 1 - 2\frac{(\xi_0^2 - \xi^2)(\eta_0^2 - \eta^2)}{(\eta^2 - \xi^2)(\eta_0^2 - \xi_0^2)}. \tag{7.5}$$

The domain Ω is the union of the triangle Δ_{yx} and the rectangle Λ_{xy} with vertices (x, y), $(0, y - x)$, $(y - x, 0)$ and (y, x).

When (s, t) belongs to Λ_{xy}, we have

$$0 \le |\xi_0| \le \xi \le \eta_0 \le \eta < +\infty,$$

and the formula (7.5) shows that $0 \le z \le 1$. We derive from the properties of P_l (see [14]) that there exists a positive constant M_1 such that

$$|P_{\alpha - \frac{1}{2}}(1 - 2z)| \le M_1, \; (s, t) \in \Lambda_{xy}.$$

When (s, t) is in Δ_{yx} we remark that x_1 is positive while x_2 is negative, so the formula (7.5) shows that

$$1 \le 1 - 2z \le 1 - 2x_2,$$

and on the other hand

$$0 \le -x_2 \le \frac{y}{t},$$

so using the integral representation of P_l we have

$$|P_l(1 - 2z)| \le (1 - 2x_2)^l \le (\frac{6y}{t})^l.$$

The previous estimates and the properties of the domain Ω yield

$$||\mathcal{R}(x, y; s, t)|| \le M_1'(\frac{s}{x})^{\alpha + \frac{1}{2}}, \; (s, t) \in \Omega, \tag{7.6}$$

where M_1' is a given positive constant.

Moreover, using the previous estimates, we have (see [2])

$$\begin{aligned} &\lim_{t \to 0^+} \mathcal{R}(x, y; s, t) = 0, \\ &\lim_{t \to 0^+} \tfrac{1}{2}[-\tfrac{\partial}{\partial t}\mathcal{R}(x, y; s, t) + \tfrac{2\alpha+1}{t}\mathcal{R}(x, y; s, t)] = \omega_0(x, y, s)I, \end{aligned} \tag{7.7}$$

where

$$\omega_0(x, y, s) = \frac{2^{1-2\alpha}\Gamma(\alpha + 1)}{\sqrt{\pi}\Gamma(\alpha + \frac{1}{2})} s(xy)^{-2\alpha} W^{\alpha - \frac{1}{2}}(x, y, s), \tag{7.8}$$

with

$$W(x, y, s) = \{(x + y)^2 - s^2\}\{s^2 - (x - y)^2\}. \tag{7.9}$$

7.2. Solution in the case $0 < y \leq x$. For any fixed point $(x, y) \in \mathbb{R}^2,\ 0 < y \leq x$, and $(s, t) \in \Delta_{xy}$, let $v(s, t) = \mathcal{R}(x, y; s, t)$ be the solution of (7.1) and $u(s, t)$ the solution of the problem (1.3). We have

$$\begin{aligned} v(s,t)[(L_s^\alpha - L_t^\alpha)u(s,t)] - [(L_s^\alpha - L_t^\alpha)^{**}v(s,t)]u(s,t) \\ = u(s,t)q^*(t) - q(s)u(s,t). \end{aligned} \tag{7.10}$$

We apply Green's theorem to the expression in the left on the triangle Δ_{xy} (see [7]); then we deduce that the function $u(s, t)$ is a solution of the integral equation

$$u(x,y) = \int_{x-y}^{x+y} \omega_0(x,y,s)f(s)ds + \frac{1}{2}\int_{\Delta_{xy}} \mathcal{R}(x,y;s,t)[q(s)u(s,t) - u(s,t)q^*(t)]dsdt.$$

This integral equation is of Volterra type, and therefore can be solved by the successive approximations method. For this purpose we set

$$u_0(x,y) = \int_{|x-y|}^{x+y} \omega_0(x,y,s)f(s)ds, \tag{7.11}$$

and, for $k \geq 0$,

$$u_{k+1}(x,y) = \frac{1}{2}\int_{\Delta_{xy}} \mathcal{R}(x,y;s,t)[q(s)u_k(s,t) - u_k(s,t)q^*(t)]dsdt. \tag{7.12}$$

Theorem 7.1. *Let $0 < y \leq x$, and f be a bounded measurable function. Suppose that there exists a measurable function $\mathcal{Q}(t)$ such that*

$$||q(s)|| + ||q(t)|| \leq \mathcal{Q}(t), \qquad s \geq t,$$

and $\rho_0(y) = \int_0^y \mathcal{Q}(t)dt$ is finite; then the series

$$u(x,y) = \sum_{k \geq 0} u_k(x,y)$$

is uniformly convergent in any domain

$$\Sigma_\delta = \{(x,y) \in \mathbb{R}^2 \mid y \leq x, \text{ and } 0 < y \leq \delta\},\ \delta < +\infty.$$

The sum $u(x, y)$ is a solution of the problem (1.3), and there exist positive constants M and N such that

$$||u(x,y)|| \leq N||f||_\infty exp\{My\rho_0(y)\}.$$

Proof. Using (7.8) and (7.11) we deduce that there exists a positive constant N_0 such that

$$||u_0(s,t)|| \leq N_0||f||_\infty. \tag{7.13}$$

Then the hypothesis of the theorem, formulas (7.4) and (7.12) show recursively that

$$||u_k(s,t)|| \leq N_0||f||_\infty \frac{[M_0' y \rho_0(y)]^k}{k!}, \qquad k \geq 1,$$

so Theorem 7.1 follows. □

7.3. The translation kernel. In this part we suppose that $0 < y \leq x$ and that q satisfies the hypothesis of Theorem 7.1.

Any $M_n(\mathbb{C})$-valued function f can be written as

$$f(x) = \sum_{1\leq i,j\leq n} \delta^{ij}(x) f_{ij}(x),$$

where $\delta^{ij},\ 1 \leq i, j \leq n$, is the matrix-valued function defined by

$$\delta^{ij}_{kl} = \begin{cases} 1 & \text{if} \quad (k,l) = (i,j), \\ 0 & \text{if} \quad (k,l) \neq (i,j), \end{cases}$$

and f_{ij} is a scalar-valued function.

Let i and $j \in \mathbb{N}$, $1 \leq i, j \leq n$, be two fixed integers and consider the following Cauchy problem:

$$\begin{cases} u_{xx} + \frac{2\alpha+1}{x} u_x - u_{yy} - \frac{2\alpha+1}{y} u_y = uq^*(y) - q(x)u, \\ u(x,0) = \delta^{ij} h(x),\ \frac{\partial u}{\partial y}(x,0) = 0, \end{cases} \tag{7.14}$$

where h is a bounded, measurable and scalar-valued function.

Let $u(x, y)$ be the solution of (7.14). We shall show that each function $u_k(x, y)$, $k \geq 0$, in (7.12) may be represented by

$$u_k(x,y) = \int_{x-y}^{x+y} w_k^{ij}(x,y,\eta) h(\eta) d\eta,$$

where, particularly, we have

$$w_0^{ij}(x,y,\eta) = \omega_0(x,y,\eta)\delta^{ij}.$$

Let

$$\tilde{w}^{ij}(x,y,\eta) = \sum_{k\geq 1} w_k^{ij}(x,y,\eta);$$

then we have the following results.

Proposition 7.2. *Under the hypothesis of Theorem 7.1 the series*

$$w^{ij}(x,y,\eta) = w_0^{ij}(x,y,\eta) + \tilde{w}^{ij}(x,y,\eta) \tag{7.15}$$

is uniformly convergent on any compact region of Σ_δ and satisfies the estimates

1. $||\tilde{w}^{ij}(x,y,\eta)|| \leq N\eta(xy)^{-1}[-1 + expM\rho_1(y)]\ ,\ \frac{1}{2} \leq \alpha,$
2. $||\tilde{w}^{ij}(x,y,\eta)|| \leq N\omega_0(x,y,\eta)[-1 + expM\rho_1(y)],\ 0 < \alpha < \frac{1}{2},$
3. $||\tilde{w}^{ij}(x,y,\eta)|| \leq N\eta^{\frac{3}{2}(\alpha+\frac{1}{2})} x^{|\alpha|-\frac{1}{2}} y^{\frac{|\alpha|}{2}-\frac{5}{4}}[-1 + expMy\rho_0(y)],\ -\frac{1}{2} < \alpha < 0,$
4. $||\tilde{w}^{ij}(x,y,\eta)|| \leq N\eta^{3/4} x^{-1/2} y^{-1/4} \rho_0(y) exp[My\rho_0(y)],\ \alpha = 0,$

where N, M are positive constants depending on α, $\rho_\epsilon(y) = \int_0^y t^\epsilon \mathcal{Q}(t)dt$, and $\epsilon > 0$.

Proof. From (7.11) we obtain

$$u_0(x,y) = \int_{x-y}^{x+y} w_0^{ij}(x,y,\eta) h(\eta) d\eta.$$

Using the previous formula and changing the order of integration we rewrite (7.12), for $k = 0$, in the form (see [15])

$$u_1(x,y) = \frac{1}{2}\int_{x-y}^{x+y}\int_{\sigma_\eta} \mathcal{R}(x,y;s,t)[q(s)w_0^{ij}(s,t,\eta) - w_0^{ij}(s,t,\eta)q^*(t)]h(\eta)dsdtd\eta,$$

where σ_η is the rectangle with vertices (x,y), $(\frac{\eta+x-y}{2},\frac{\eta-x+y}{2})$, $(\eta,0)$ and $(\frac{\eta+x+y}{2},\frac{x+y-\eta}{2})$.

Recursively we have

$$u_k(x,y)=\int_{x-y}^{x+y} w_k^{ij}(x,y,\eta)h(\eta)d\eta,\ k\geq 1,$$

with

$$w_{k+1}^{ij}(x,y,\eta)=\frac{1}{2}\int_{\sigma_\eta}\mathcal{R}(x,y;s,t)[q(s)w_k^{ij}(s,t,\eta)-w_k^{ij}(s,t,\eta)q^*(t)]dsdt. \tag{7.16}$$

For the estimates of $w_k^{ij}(x,y,\eta)$ we use again the results of [2].

Since for $\alpha\geq\frac{1}{2}$ there exists a positive constant M_2 such that

$$||\mathcal{R}(x,y;s,t)\omega_0(s,t,\eta)||\leq M_0M_2\ \eta(xy)^{-1},\ (s,t)\in\Delta_{xy}.$$

Hence by (7.16) we show, recursively, that

$$||w_k^{ij}(x,y,\eta)||\leq M_2\eta(xy)^{-1}\ \frac{[M_0\rho_1(y)]^k}{k!},\ k\geq 1;$$

then we deduce (i).

In case of $0<\alpha<\frac{1}{2}$, we have $Z\leq 16xystW^{-1}(x,y,\eta)$, so by (7.8) we have

$$||\omega_0(s,t,\eta)\mathcal{R}(x,y;s,t)||\leq M_2\eta(st)W^{-1/2}(s,t,\eta)\omega_0(x,y,\eta).$$

Then, using the substitution $s^2=(\eta-t)^2+4\eta tv$, we deduce that

$$W(s,t,\eta)=16t^2\eta^2v(1-v), \tag{7.17}$$

and we show, recursively, that there exists a positive constant M_2 such that

$$||w_k^{ij}(x,y,\eta)||\leq M_2\omega_0(x,y,\eta)\frac{[M_0\rho_1(y)]^k}{k!},\ k\geq 1;$$

from this we deduce (ii).

In the case $-\frac{1}{2}<\alpha<0$, according to (7.2) and (7.8), we have

$$||\omega_0(s,t,\eta)\mathcal{R}(x,y;s,t)||\leq M_0M_2(st)^{-\alpha+\frac{1}{2}}\eta(xy)^{-\alpha-\frac{1}{2}}W^{\alpha-\frac{1}{2}}(s,t,\eta);$$

using formulas (7.16) and (7.17), we show that

$$||w_k^{ij}(x,y,\eta)||\leq M_2x^{|\alpha|-1/2}\eta^{\frac{3}{2}(\alpha+\frac{1}{2})}y^{\frac{1}{2}|\alpha|-\frac{5}{4}}\frac{[M_0y\rho_0(y)]^k}{k!},\ k\geq 1\ .$$

We discuss finally the case $\alpha=0$. Since we have $\omega_0(s,t,\eta)=M_\alpha\eta W^{-\frac{1}{2}}(s,t,\eta)$, using (7.2), we deduce that

$$||w_1^{ij}(x,y,\eta)||\leq M_0M_2\eta(xy)^{-\frac{1}{2}}\int_0^y t^{\frac{1}{2}}\mathcal{Q}(t)\int_{\eta-t}^{\eta+t}s^{\frac{1}{2}}W^{-\frac{1}{2}}(s,t,\eta)dsdt.$$

By the same substitution as the previous cases and thanks to (7.17) we deduce that

$$||w_1^{ij}(x,y,\eta)||\leq M_0M_2\eta^{\frac{3}{4}}(xy)^{-\frac{1}{2}}\rho_{\frac{1}{4}}(y).$$

Then, recursively, we obtain

$$||w_k^{ij}(x,y,\eta)||\leq M_2\eta^{\frac{3}{4}}x^{-\frac{1}{2}}y^{-\frac{1}{4}}y^{k-1}\frac{[M_0\rho_0(y)]^k}{k!};$$

this completes the proof of the proposition. □

Theorem 7.3. *In case of* $0 < y \leq x$ *and under the hypothesis of Theorem 7.1 the function*

$$u(x,y) = \sum_{i,j=1}^{n} \int_{x-y}^{x+y} w^{ij}(x,y,\eta) f_{ij}(\eta) d\eta$$

is the solution of the Cauchy problem (1.3).

7.4. Solution in the case $0 < x \leq y$**.** In this case we introduce the domains $\Omega_1 = \Omega \cap \{y \leq x\}$ and $\Omega_2 = \Omega \cap \{x \leq y\}$ (see Figure 2, [8]). We apply again the Riemann method in the domain Ω. After computation, we show that Ω is a characteristic domain in the same way as Δ_{xy}. Indeed, for $v(s,t) = \mathcal{R}(x,y;s,t)$, we have

$$\left\{ \begin{array}{ll} (L_s^\alpha - L_t^\alpha)^{**} v(s,t) = 0, & (s,t) \in \Omega, \\ v_s + v_t = (\alpha + \frac{1}{2})(\frac{1}{s} + \frac{1}{t})v, & s - t = x - y, \\ v_s - v_t = (\alpha + \frac{1}{2})(\frac{1}{s} - \frac{1}{t})v, & s + t = x + y, \\ v_s - v_t = (\alpha + \frac{1}{2})(\frac{1}{s} - \frac{1}{t})v, & s + t = y - x, \\ v(x,y) = 1. & \end{array} \right. \tag{7.18}$$

Considering the formula (7.10), applying Green's theorem to the expression in the left on the domain Ω and using (7.18), we show that the solution $U(x,y)$ of the Cauchy problem (1.3) satisfies, for $0 < x \leq y$, the integral equation

$$U(x,y) = u_0(x,y) + \frac{1}{2} \int_\Omega \mathcal{R}(x,y;s,t)[q(s)U(s,t) - U(s,t)q^*(t)] dsdt \tag{7.19}$$

where $u_0(x,y)$ is given by (7.11).

Theorem 7.4. *Let* $0 < x \leq y$*; under the hypothesis of Theorem 7.1, the problem (1.3) has an unique solution* $U(x,y)$ *satisfying*

$$||U(x,y)|| \leq N ||f||_\infty (\frac{x+y}{2x})^{\alpha + \frac{1}{2}} exp[2My\rho_0(y)],$$

where N, *and* M *are given positive constants.*

Proof. Let

$$v_0(x,y) = \frac{1}{2} \int_{\Omega_1} \mathcal{R}(x,y;s,t)[q(s)u(s,t) - u(s,t)q^*(t)] dsdt;$$

since, by Theorem 7.1, $u(x,y)$ is completely defined on Ω_1 so is $v_0(x,y)$, and by (7.6) we have

$$||v_0(x,y)|| \leq N ||f||_\infty y \rho_o(y) (\frac{y+x}{x})^{\alpha + \frac{1}{2}} exp My\rho_0(y). \tag{7.20}$$

Then we rewrite (7.19) in the form

$$U(x,y) = u_0(x,y) + v_0(x,y) + \frac{1}{2} \int_{\Omega_2} \mathcal{R}(x,y;s,t)[q(s)U(s,t) - U(s,t)q^*(t)] dsdt.$$

Solving this problem causes us to consider the two following integral equations:

$$\tilde{U}(x,y) = u_0(x,y) + \frac{1}{2} \int_{\Omega_2} \mathcal{R}(x,y;s,t)[q(s)\tilde{U}(s,t) - \tilde{U}(s,t)q^*(t)] dsdt$$

and

$$\tilde{\tilde{U}}(x,y) = v_0(x,y) + \frac{1}{2} \int_{\Omega_2} \mathcal{R}(x,y;s,t)[q(s)\tilde{\tilde{U}}(s,t) - \tilde{\tilde{U}}(s,t)q^*(t)] dsdt.$$

To solve the previous integral equations we use successive approximations. By means of Theorem 7.1, the formulas (7.6) and (7.13) we deduce that there exists a positive constant N_1 such that

$$\tilde{U}(x,y) \leq N_1||f||_\infty(\frac{y+x}{2x})^{\alpha+\frac{1}{2}}exp[My\rho_0(y)].$$

In a similar manner, using (7.6) and (7.20), we have

$$\tilde{\tilde{U}}(x,y) \leq N_1||f||_\infty(\frac{y+x}{x})^{\alpha+\frac{1}{2}}exp[My\rho_0(y)][-1+expMy\rho_0(y)].$$

Then $\tilde{U}+\tilde{\tilde{U}} = U$ is a solution of (7.19), easily, we have Theorem 7. □

7.5. The product formula. The following result generalizes the product formula studied extensively in the scalar case (see [5], [7]).

Theorem 7.5. *For any $\lambda \in \mathbb{C}$ we have*

1. $\psi(\lambda,x)\psi^*(\bar{\lambda},y) = \sum_{1\leq i,j\leq n}\int_{x-y}^{x+y} w^{ij}(x,y,\eta)\psi^{ij}(\lambda,\eta)d\eta,\ 0<y\leq x,$
2. $\psi(\lambda,x)\psi^*(\bar{\lambda},y) = \sum_{1\leq i,j\leq n}\int_{y-x}^{y+x} [w^{ij}(y,x,\eta)]^*\overline{\psi^{ij}(\bar{\lambda},\eta)}d\eta,\ 0<x\leq y,$

where $w^{ij}(x,y,\eta)$ is given by the formula (7.15).

Proof. Theorem 3.5 and Corollary 3.3 show that, when λ is real, the eigenfunction $\psi(\lambda,x)$ is bounded with respect to x so we can apply the results of Theorem 7.3. Since

$$\psi(\lambda,x) = \sum_{1\leq i,j\leq n}\delta^{ij}\psi^{ij}(\lambda,x),$$

the Cauchy problem (1.3), with initial conditions

$$u(x,0) = \psi(\lambda,x), \tag{7.21}$$

has, when $0<y\leq x$, the following solution:

$$u(x,y) = \sum_{1\leq i,j\leq n}\int_{x-y}^{x+y} w^{ij}(x,y,\eta)\psi^{ij}(\lambda,\eta)d\eta, \qquad \lambda\in\mathbb{R}.$$

On the other hand $u(x,y) = \psi(\lambda,x)\psi^*(\bar{\lambda},y)$ is a solution of the Cauchy problem (1.3) with the initial conditions (7.21). The unicity of the solution, and the analyticity of the function $\lambda \to \psi(\lambda,x)$ give us the theorem. □

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